Scaling and efficiency determine the irreversible evolution of a market

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In setting up a stochastic description of the time evolution of a financial index, the challenge consists in devising a model compatible with all stylized facts emerging from the analysis of financial time series and providing a reliable basis for simulating such series. Based on constraints imposed by market efficiency and on an inhomogeneous-time generalization of standard simple scaling, we propose an analytical model which accounts simultaneously for empirical results like the linear decorrelation of successive returns, the power law dependence on time of the volatility autocorrelation function, and the multiscaling associated to this dependence. In addition, our approach gives a justification and a quantitative assessment of the irreversible character of the index dynamics. This irreversibility enters as a key ingredient in a novel simulation strategy of index evolution which demonstrates the predictive potential of the model.

complex systems | finance | stochastic processes

For over a century, it has been recognized (1) that the unpredictable time evolution of a financial index is inherently a stochastic process. However, despite many efforts (2–11), a unified framework for simultaneously understanding empirical facts (12–19), such as the non-Gaussian form and multiscaling in time of the distribution of returns, the linear decorrelation of successive returns, and volatility clustering, has been elusive. This situation occurs in many natural phenomena, when strong correlations determine various forms of anomalous scaling (20–27). Here, by employing mathematical tools at the basis of a generalization of the central limit theorem to strongly correlated variables (28), we propose a model of index evolution and a corresponding simulation strategy which account for all robust features revealed by the empirical analysis.

Let $S(t)$ be the value of a given asset at time t. The logarithmic return over the interval $[t, t+T]$ is defined as $r(t, T) \equiv \ln S(t+T) - \ln S(t)$, where $t = 0, 1, \ldots$ and $T = 1, 2, \ldots$, in some unit (e.g., day). From a sufficiently long historical series, one can sample the empirical probability density function (PDF) of r over a time T, $\bar{p}_T(r)$, and the joint PDF of two successive returns $r_1 \equiv r(t, T)$ and $r_2 \equiv r(t+T, T)$, denoted by $\bar{p}^{(2)}_{2T}(r_1, r_2)$. This joint PDF contains the information on the correlation between r_1 and r_2 in the sampling. A well established property (13–16) is that, if T is longer than tens of minutes, the linear correlation vanishes: $\int \bar{p}^{(2)}_{2T}(r_1, r_2) r_1 r_2\, dr_1 dr_2 \equiv \langle r_1 r_2 \rangle_{\bar{p}^{(2)}_{2T}} = 0$. This is a consequence of the efficiency of the market (3), which quickly suppresses any arbitrage opportunity. Another remarkable feature is that, within specific T ranges, $\bar{p}_T$ approximately assumes a simple scaling form

$$\bar{p}_T(r) = \frac{1}{T^{\bar{D}}}\, \bar{g}\left(\frac{r}{T^{\bar{D}}}\right), \quad [1]$$

where $\bar{g}$ and $\bar{D}$ are the scaling function and exponent, respectively. Eq. **1** manifests self-similarity, a symmetry often met in natural phenomena (20–23, 27): plots of $T^{\bar{D}}\bar{p}_T$ vs. $r/T^{\bar{D}}$ for different T values collapse onto the same curve representing $\bar{g}$. We verify the scaling ansatz in Eq. **1** for the Dow Jones Industrial (DJI) index using a dataset of more than one century (1900–2005) of daily closures. This index is paradigmatic of market behavior and the considerable number of data reduces sampling fluctuations substantially. In Fig. 1 the collapse of the empty symbols is rather satisfactory (the explanation of the meaning of the full symbols in Fig. 1 is given below). The scaling function in Eq. **1** is non-Gaussian (2, 12–17). Although linear correlations vanish, in the T range considered $\bar{g}$ is determined by the strong *nonlinear* correlations of the returns. Only for $T > \tau_c$ (with τ_c of the order of the year) successive index returns become independent and $\bar{p}_T$ turns Gaussian in force of the central limit theorem (29).

Results and Discussion

Our first goal is to establish up to what extent the assumption of simple scaling in Eq. **1** does constrain the structure of the joint PDF $\bar{p}^{(2)}_{2T}$. One must of course have

$$\int \bar{p}^{(2)}_{2T}(r_1, r_2)\delta(r - r_1 - r_2)dr_1 dr_2 = \bar{p}_{2T}(r),$$
$$\int \bar{p}^{(2)}_{2T}(r_1, r_2)dr_2 = \bar{p}_T(r_1), \quad [2]$$
$$\int \bar{p}^{(2)}_{2T}(r_1, r_2)dr_1 = \bar{p}_T(r_2).$$

Indeed, the first line of Eqs. **2** follows from $r(t, 2T) = r_1 + r_2$. Furthermore, because the joint PDF $\bar{p}^{(2)}_{2T}$ is sampled from a sequence of time-translated intervals of duration $2T$ along the historical series, both the first and the second halves of all such intervals provide an adequate sampling basis for $\bar{p}_T$. This justifies the second and third lines of Eqs. **2**. At this point, we notice that the property $\langle r_1 r_2 \rangle_{\bar{p}^{(2)}_{2T}} = 0$ implies that $\langle r^2 \rangle_{\bar{p}_{2T}} = \langle (r_1 + r_2)^2 \rangle_{\bar{p}^{(2)}_{2T}} = 2\langle r^2 \rangle_{\bar{p}_T}$. In force of Eq. **1**, $\langle r^2 \rangle_{\bar{p}_T} \sim T^{2\bar{D}}$. Hence, we obtain $2T^{2\bar{D}} = (2T)^{2\bar{D}}$, i.e., $\bar{D} = 1/2$. Remarkably, for all developed market indices, $\langle r^2 \rangle_{\bar{p}_T}$ is found to scale consistently with a $\bar{D}$ pretty close to 1/2 (30).

By switching to Fourier space in Eq. **2**, the notion of a generalized product operation allows to identify a solution for $\bar{p}^{(2)}_{2T}$ in terms of $\bar{p}_T$ alone. Although the ordinary multiplication of characteristic functions (i.e., Fourier transforms) of Gaussian PDFs would yield trivially the correct $\bar{p}^{(2)}_{2T}$ in the case of independent successive returns (29), the generalized product is used here to take into account strong nonlinear correlations consistently with the anomalous scaling they determine [see

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ECONOMIC SCIENCES

STATISTICS

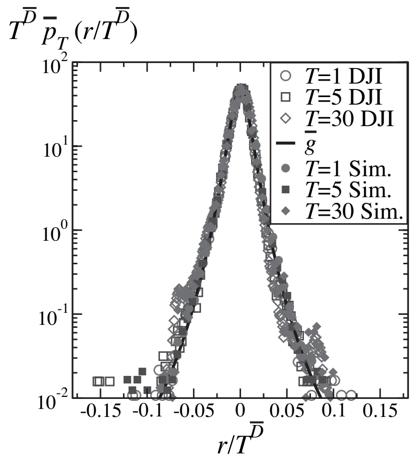

Fig. 1. Data collapse for $\bar{p}_T(r)$ (T measured in days) sampled from a record of $\approx 2.7 \times 10^4$ DJI daily closures (open symbols). The average daily trend of the order of 10^{-4} has been subtracted. The collapse analysis furnishes the scaling function $\bar{g}$ reported as the full line and the scaling exponent $\bar{D} \simeq 1/2$ (see also Fig. 3 and *SI Text*). Filled symbols report the data collapse for the results of a single simulation of the DJI history.

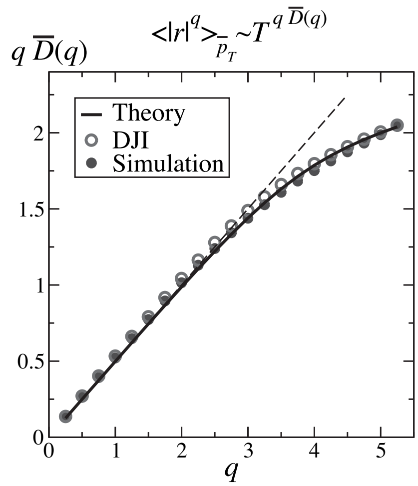

Fig. 3. Scaling exponent of the qth moment of $\bar{p}_T$. Open (filled) circles refer to the DJI data (simulation) of Fig. 1. The dashed line is $q/2$. Multiscaling is due to the deviation of $\bar{D}(q)$ from a constant value. The full line reports the time-averaged asymptotic ($\tau_c \gg 1$) theoretical prediction based on Eq. **5**.

supporting information (SI) *Text*] and can be seen to be at the basis of a central limit theorem (28). Our solution is strongly supported by the remarkable consistency with the numerical results and by the analogy with the independent case. In Fig. 2 we compare the PDF of the return r_2 conditioned to a given absolute value of the return r_1, as obtained through our solution (continuous lines), with the empirically sampled one (symbols). The agreement does not involve fitting parameters, because $\bar{D}$ and those entering the assumed analytical form of $\bar{g}$ are already fixed in Fig. 1.

At this point, we must take into account that the simple scaling ansatz in Eq. **1** is only approximately valid (7). Indeed, a consequence of Eq. **1** is $\langle |r|^q \rangle_{\bar{p}_T} \sim T^{q\bar{D}}$, which we exploited above for $q = 2$. However, a careful analysis reveals that the qth moment exponent deviates from the linear behavior $q\bar{D} \simeq q/2$ for $q \gtrsim 3$ (open circles in Fig. 3). Like the linear behavior with slope 1/2 observed for low-order moments, this multiscaling effect is common to most indices (30). To explain this feature, we have to investigate the relation between the empirical $\bar{p}_T$ and the stochastic process generating the time series. If PDFs like $\bar{p}_T$ and $\bar{p}^{(2)}_{2T}$ were directly describing such a process, this would be with stationary increments. This assumption is legitimate only for sufficiently long times, larger than τ_c. Below, we identify in the interplay between scaling and nonstationarity a precise mechanism accounting for the robust features of $\bar{p}_T$ detected for $T \ll \tau_c$, including its multiscaling.

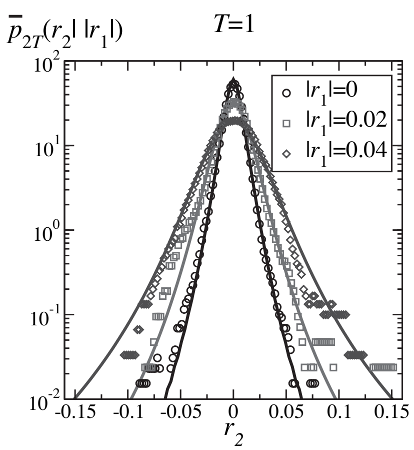

Fig. 2. Conditional probabilities of daily returns r_2 for different values of r_1. Empty symbols refer to the DJI data. The continuous curves are the predictions of our theory for $\bar{p}^{(2)}_{2T}(r_2 \| r_1|) = [\bar{p}^{(2)}_{2T}(r_1,r_2) + \bar{p}^{(2)}_{2T}(-r_1,r_2)]/\int[\bar{p}^{(2)}_{2T}(r_1,r_2) + \bar{p}^{(2)}_{2T}(-r_1,r_2)]dr_2$. The absolute value of r_1 is introduced for reducing sample fluctuations.

Let us indicate by $p_{t,T}$ and $p^{(2)}_{t,2T}$ the ensemble PDF's corresponding to $\bar{p}_T$ and $\bar{p}^{(2)}_{2T}$, respectively. The additional dependence on t, the initial time of the interval $[t, t + T]$, shows that we do not assume stationarity for these PDF's. We postulate that, within specific T ranges (e.g., the one in Fig. 1), $p_{0,T}$ obeys a simple scaling like that in Eq. **1**, but possibly with a D and a g different from $\bar{D}$ and $\bar{g}$, respectively. One then realizes that this scaling and the linear decorrelation of returns impose on $p^{(2)}_{t,2T}$ constraints analogous to those for $\bar{p}^{(2)}_{2T}$ in Eq. **2**, except for the third one, which now reads

$$\int p^{(2)}_{0,\,2T}(r_1, r_2)dr_1 \equiv p_{T,\,T}(r_2) = p_{0,\,aT}(r_2). \qquad \textbf{[3]}$$

This last condition tells us that, as a consequence of the nonlinear correlations, the effective time span of the marginal PDF obtained by integrating $p^{(2)}_{0,2T}$ in r_1 must be renormalized by a factor a. This factor is determined again by consistency of the second moments scaling properties, as above. Since now $\langle |r|^2 \rangle_{p_{0,T}} \sim T^{2D}$, one gets from Eq. **3** $a = (2^{2D} - 1)^{1/2D}$. So, $D \neq 1/2$ implies $a \neq 1$, and thus nonstationarity and irreversibility of the process. Similar functional relations hold for the PDFs of the magnetization of critical spin models upon doubling the system size and can be explained in that context by the renormalization group theory (27). Our generalized multiplication of characteristic functions allows us to express $p^{(2)}_{t,2T}$ in terms of $p_{t,T}$ and to establish the time-inhomogeneous scaling property

$$p_{t,\,T}(r) = \frac{1}{\sqrt{(t+T)^{2D} - t^{2D}}}\, g\left(\frac{r}{\sqrt{(t+T)^{2D} - t^{2D}}}\right). \qquad \textbf{[4]}$$

It remains now to make explicit the link between the p values and the sampled $\bar{p}$, and to determine D. By construction, $\bar{p}_T$ is a t average of $p_{t,T}$. Because the time inhomogeneity of $p_{t,T}$ must cross over into homogeneity for t exceeding τ_c, we expect the following approximation

$$\bar{p}_T(r) = \frac{1}{\tau_c} \sum_{t=0}^{\tau_c - 1} p_{t,T}(r) \qquad [5]$$

to hold. Indeed, the history over which $\bar{p}_T$ is sampled is much longer than τ_c and allows in principle also an indirect sampling of $p_{t,T}$ if we simply assume $p_{t+\tau_c,T}(r) = p_{t,T}(r)$.

Despite the fact that Eq. **4** implies a simple scaling exponent D for $p_{0,T}$, Eq. **5** leads to the remarkable property that, independently of D, the low-q moments of $\bar{p}_T$ approximately scale with exponent $q/2$ as soon as $\tau_c \gg 1$. Moreover, if $D < 1/2$, $\bar{p}_T$ displays a multiscaling of the same type as that found empirically: $\bar{D}(q) < 1/2$ for the high-order moments. The matching of the theoretical predictions for the multiscaling of $\bar{p}_T$ on the basis of Eq. **5** with the empirical results is a first way of identifying D. For the DJI, in Fig. 3 we show that with $D = 0.24$ this matching is very satisfactory. The scaling functions of $p_{0,T}$ and $\bar{p}_T$ can also be shown to be simply related, once D is known. We notice that the observed multiscaling features of financial indices, which inspired multiplicative cascade models (8, 19) in analogy with turbulence (20, 21), are explained here in terms of an additive process possessing the time-inhomogeneous scaling (Eq. **4**).

The introduction of autoregressive schemes like ARCH (5) marked an advance in econometrics and financial analysis (6, 9), and, more generally, in the theory of stochastic processes. In an autoregressive simulation, a number of parameters weighting the influence of the past history on the PDF of the following return must be fixed through some optimization procedure. By our approach, a generalization of $p^{(2)}_{t,2T}$ to the case of n consecutive intervals can be fully expressed just in terms of $p_{t,T}$ and D. This is obtained by taking the inverse Fourier transform of our solution for the characteristic function of the joint PDF (see *SI Text*). In this way, we can precisely calculate the PDF that rules the extraction of the ith return, r_i, giving us as conditioning inputs the previous m ones, $r_{i-m}, \ldots, r_{i-1}$. Consistently with our schematization in Eq. **5**, the existence of exogenous factors acting on the market can be taken into account by resetting the width of the marginal PDFs with an (average) periodicity equal to τ_c (see *SI Text*). The results for a single simulation with $m = 100$, $\tau_c = 500$, and $D = 0.24$ are illustrated by the filled symbols in Figs. 1 and 3. The coincidence of the scaling properties observed for the DJI with those of our simulation furnish a second strong indication of the validity of our approach and of the estimation of D.

The correctness of the value of D can be further checked by considering the volatility autocorrelation function at time separation τ (Fig. 4). A well established fact (12–16, 19) is its power law decay $c(\tau) \approx \tau^{-\beta}$ for $\tau < \tau_c$, with $\beta \simeq 0.2$ for the DJI. This behavior is not reproduced by routine simulation methods in quantitative finance like GARCH (6) and requires the introduction of more sophisticated, fractional integration techniques (9). The full characterization of the joint PDF of n-consecutive returns allows us to obtain a model expression for $c(\tau)$, which

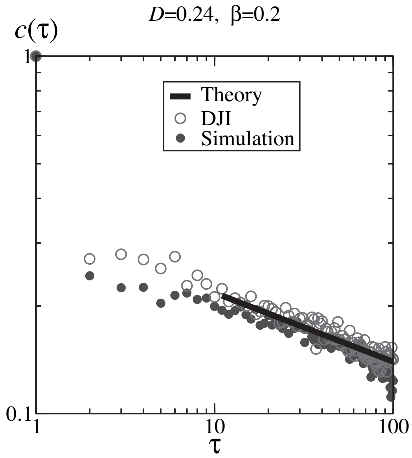

Fig. 4. Volatility autocorrelation at time separation τ (in days), $c(\tau) \equiv (\sum_{t=0}^{t_{max}} |r(t,1)||r(t+\tau,1)| - \sum_{t=0}^{t_{max}} |r(t,1)| \sum_{t=0}^{t_{max}} |r(t+\tau,1)|/t_{max}) / \sum_{t=0}^{t_{max}} |r(t,1)|^2 - [\sum_{t=0}^{t_{max}} |r(t,1)|]^2/t_{max}$ where $t_{max} + \tau - 1$ is the total length of the time series. Symbols are as in Fig. 3. The simulation here is precisely the same we refer to in Figs. 1 and 3. The full line gives the slope of the time-averaged asymptotic ($\tau_c \gg 1$, $\tau \lesssim \tau_c$) model prediction for the volatility autocorrelation, superimposed to the data (see *SI Text*).

again takes into account the nonstationarity of the process (see *SI Text*). Such an expression behaves asymptotically as a power of τ with an exponent depending on D [$c(\tau)$ is constant for $D = 1/2$ and decays for $D < 1/2$]. In particular, with $D = 0.24$ both the model asymptotic expression and the results of our simulation procedure furnish a nice agreement with the exponent $\beta \simeq 0.2$ observed for the DJI index (Fig. 4). Thus, the algebraic volatility autocorrelation function decay is reproduced by our scheme and provides a second criterion to fix consistently the anomalous scaling exponent D.

Our approach is based on two postulates: inhomogeneous-time scaling and the vanishing of linear return correlations. These symmetries lead, in an unambiguous, deductive manner, to a model for the underlying stochastic process determining market evolution. Of course, the results follow only when the postulates are valid and we have shown that within specific time-ranges the consequences of these postulates are in remarkable agreement with the data. Major advances in understanding critical phenomena worked in a similar vein decades ago (20), when the scaling assumptions allowed to establish links between seemingly disparate phenomena and put the basis for the development of renormalization group theory (27). So far, the coexistence of anomalous scaling with the requirement of absence of linear correlation imposed by economic principles has been regarded as an outstanding open problem in the theory of stochastic processes. We believe that our solution could be relevant for developments in this field, as well as for describing scaling behaviors of other complex systems (20–26).

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